Daya Assessment
Admissions Testing

Not actually a Sixth Term Examination Paper **1312**

MATHEMATICS D Morning

Thursday Time: 3 hours

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so, or do what you want; I ain't the boss of you

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet, so that the examiner knows who you are, because I don't. I'm kidding, I'm kidding... there's no examiner, mark it yourself you twit.

INFORMATION FOR CANDIDATES

There are 12 questions in this paper.
Each question is marked out of 20, but I didn't allocate marks. There is no restriction of choice. But I recommend doing all of it since some of it is dead easy.

No questions attempted will be marked. The same goes for the unattempted

Your final mark could based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on way more than **six** questions. Little credit will be given for fragmentary answers, since they kinda suck.

You must shade the appropriate Question Answered circle on every page of the answer booklet that you write on. Failure to do so might mean that some of your answers are not marked. I'm kidding, nothing will be marked

There is NO Mathematical Formulae Booklet, you brain-dead git.

Calculators are not permitted as they are weapons of math destruction.

Wait to be told you may begin before turning this page. You may have to wait a bit so just do it when you're ready.

Section A: Pure Mathematics

1 Show graphically, or otherwise, that a curve $y = \mathrm{f}(x)$ enlarged by a scale factor k about the origin is defined by:

$$kf\left(\frac{x}{k}\right), \qquad (*)$$

for $k, x \in \mathbb{R}$

(i) Given that $ax^2 + bx + c$ is a scale of curve $\mathrm{h}(x)$ about the origin, with a scale factor a, find $\mathrm{h}(x)$

(ii) Using $(*)$, and by letting $f(x) = ax^2 + bx + c$, show that all parabolas are similar.

Show that there does not exist a value of a and a value of b such that $x^3 + bx$ and $x^3 + ax$ are similar, $a \neq b$. Further show that all polynomials of the form ax^n are similar.

2 A cubic polynomial $\mathrm{P}(x)$ has the general equation:

$$P(x) = ax^3 + bx^2 + cx + d\,,$$

Show that another way to write a cubic polynomial is in the form below:

$$P(t - n) = a(t^3 + pt + q)\,,$$

expressing p and q in terms of a,b,c & d.

(i) Given that there is only one real solution, by letting $t = u + v$ or otherwise, find the value of t that satisfies

$$0 = t^3 + pt + q\,,$$

in terms of p and q

(ii) Given that there is only one real root for $\mathrm{f}(x)$, find the roots of $\mathrm{f}(x)$ for:

(a) $f(x) = x^5 + x^4 + 1$

(b) $f(x) = x^5 + x + 1$

3 $\sigma_0(n)$ is called the *number-of-divisors function*. It outputs the number of factors of n for $n \in \mathbb{N}$. For example, $\sigma_0(12) = 6$ as 12 has 6 factors, 1, 2, 3, 4, 6 and 12.

(i) Show that if a and b are co-prime (their highest common factor is 1), that $\sigma_0(a)\sigma_0(b) = \sigma_0(ab)$

(ii) For some prime, p, find $\sigma_0(p^k)$, in terms of k for $k \in \mathbb{N}$.

(iii) Let p_i be the ith prime. Given that n can be be expressed in the form below

$$n = \prod_{i=1}^{\infty} p_i^{a_i},$$

such that a_i varies with n and the symbol $\prod$ denotes an infinite product. Find a general solution for $\sigma_0(n^k)$

(iv) How many factors does 720^4 have?

4 A regular decagon $ABCDEFGHIJ$ is inscribed in a unit circle centre O, and $AB = x$. The line AD intersects the line OB at M. Find, in terms of x, the length AM. By considering the triangle AOB, find the exact value of x and hence find the exact value of $\cos\left(\frac{\pi}{5}\right)$. Hence or otherwise, find $\cos\left(\frac{2\pi}{5}\right)$. Further find the length of the side AG

5 The Gudermannian function, $\mathrm{gd}(x)$, is defined below:

$$\mathrm{gd}(x) = \int_0^x \mathrm{sech}(t)dt.$$

(i) Evaluate the integral below:

$$\int \sec(\arctan(x))dx.$$

Hence evaluate:

$$\int \sec(\mathrm{gd}(x))dx.$$

(ii) Evaluate the integral below:

$$\int \sec(x)\,\mathrm{arsinh}(\tan(x))dx.$$

6 A right angled triangle ABC is labelled such that AB is the hypotenuse and the angle at A is smaller than that of B. Let a denote BC, b denote AC, and c denote AB. You may assume without proof that $a > b$

(i) Let the line L bisect A. By reflecting the top half of the triangle across L such that b overlaps c, show that

$$a = b\tan\left(\frac{A}{2}\right)\left(1 + \frac{1}{\sin(B)}\right).$$

(ii) By a similar approach show that

$$b\tan(A - B) + \frac{b}{\cos(A - B)} = a\,.$$

Let $0 < \beta + \alpha < \frac{\pi}{2}$. By by considering an arbitrary rectangle $ABCD$, use a similar method to show that

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\,,$$

and deduce that this must hold for all α, β.

7 Show that a reflection about a line $y = x\tan(\theta)$ is given by the matrix:

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

(i) By considering three cases for $\theta \in \mathbb{R}$, find a general equation for an invariant *curve* for the matrix

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

(ii) Find all matrices such that $y = x^2$ is an invariant curve. Generalise this for a matrix for x^n, considering two different cases for $n \in \mathbb{N}, \quad n > 1$.

(iii) Find all matrix transformations that would produce an invariant curve parameterised by

$$x = \frac{t\sqrt{3}}{2} - \frac{t^2}{2}, \qquad y = \frac{t}{2} + \frac{t^2\sqrt{3}}{2} \qquad\qquad -\infty < t < \infty$$

8 Let i, j & k are defined as three solutions to $\sqrt{-1}$. A *Quarternion* is a number made up of these roots. For example, $2 + 3i + 4j + 4k$ is a quarternion. Furthermore, quarternions do not obey commutativity.

Given that $ji = -k$ and $ij = k$, find jk and kj.

A quartenion has the general form

$$q = a + bi + cj + dk\,,$$

where $a, b, c, d \in \mathbb{R}$

(i) Given that $q^* = a - bi - cj - dk$, show that

$$q^* = -\frac{1}{2}(q + iqi + jqj + kqk)\,,$$

(ii) Write q in the form

$$z + wj\,,$$

where $z, w \in \mathbb{C}$.

Given that $qq^* = q^*q = a^2 + b^2 + c^2 + d^2$, find

$$-w^*zj + w^*z^*j + (w^*)^2\,.$$

Section B: Mechanics

9 It can be shown that the arc length of a curve between 2 points a and b is given by

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\,,$$

Derive a parametric equivalent of this formula for $x = x(r)$ and $y = y(r)$

A *tautochrone* is a type of curve defined such that an object at A takes minimal time to fall to B if it falls along the curve. One example of a tautochrone is defined below:

$$x = \theta - \sin(\theta), \qquad y = \cos(\theta) - 1 \qquad\qquad 0 \le \theta \le \pi$$

(i) Find the arc length of the curve.

(ii) By considering energy conservation, or otherwise, find the time taken for a unit mass to fall along the curve from rest.

10 Any ellipse can be generalised with the equation

$$x^2 + \frac{y^2}{b^2} = 1\,,$$

where $b \in \mathbb{R}$.

Find an expression for the eccentricity, e, of the ellipse and the value of b in terms of the c, it's focus on the positive x axis.

(i) A particle P, collides with a wall with some velocity v. Show that the angles it makes with the wall before and after the collision are both the same if and only if the particle collides elastically

(ii) Given that a particle is projected from $(c, 0)$ and collides elastically with an ellipse with foci at $(0, c)$ and $(0, -c)$, show that it passes through $(0, -c)$. immediately after the collision.

Section C: Probability and Statistics

11 Let the lifetime of a relationship, x years, from inception be modelled by the probability density function

$$f(x) = e^{-x} \, .$$

Find the expected duration of a relationship.

A relationship can be assumed to be success if it exceeds 1 year.

(i) Assuming that the end of a relationship means the start of another, find the expected number of relationships you will go through. Furthermore, find the expected time taken from you starting your first relationship to you having a relationship being a success

(ii) Let the time taken for you to start a new relationship after ending another be distributed by $xf(x)$, find the expected number of relationships you will go through. Furthermore, find the expected time taken from you starting your first relationship to you having a relationship being a success

12 Given that any root x_n of $\frac{\sin(x)}{x}$ can be factored as $\left(1 - \frac{x}{x_n}\right)$, show, by a suitable Mclaurin expansion,

$$\sum_{i=0}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \, .$$

For the remainder of the question, you may assume without proof that any number can be uniquely expressed as the product of primes

(i) Show that the probability F_2 that any pair of random integers have a factor of 2 is given by

$$F_2 = 1 - \frac{1}{4} \, ,$$

and further show that the probability F_p that a prime p is not a factor of any pair of random integers is given by

$$F_p = 1 - \frac{1}{p^2} \, ,$$

(ii) Find the probability that any random pair of integers are coprime (they have no factors in common except for 1).